Prime decomposition and correlation measure of finite quantum systems

1999 J. Phys. A: Math. Gen. 32 L63

(http://iopscience.iop.org/0305-4470/32/5/001)

J. Phys. A: Math. Gen. **32** (1999) L63–L69. Printed in the UK PII: S0305-4470(99)98662-6

LETTER TO THE EDITOR

Prime decomposition and correlation measure of finite quantum systems

D Ellinas†§ and E G Floratos‡‖

† Applied Mathematics and Computers Laboratory, Technical University of Crete, GR-73 100, Chania, Crete, Greece

‡ NCRC Demokritos, Institute of Nuclear Physics, GR-15310 Ag, Paraskevi, Attiki, Greece and Department of Physics, University of Crete, Crete, Greece

Received 26 October 1998

Abstract. Under the name prime decomposition (PD), a unique decomposition of an arbitrary N-dimensional density matrix ρ into a sum of separable density matrices with dimensions determined by the coprime factors of N is introduced. For a class of density matrices a complete tensor product factorization is achieved. The construction is based on the Chinese remainder theorem, and the projective unitary representation of Z_N by the discrete Heisenberg group H_N. The PD isomorphism is unitarily implemented and it is shown to be co-associative and to act on H_N as comultiplication. Density matrices with complete PD are interpreted as group-like elements of H_N. To quantify the distance of ρ from its PD a trace-norm correlation index $\mathcal{E}$ is introduced and its invariance groups are determined.

Quantum correlations, an emblematic notion of quantum theory, has remained an open challenge since the early days of quantum mechanics [1, 2]. Recent investigations have set important questions concerning classification of various types of quantum correlations and their appropriate quantification. These theoretical activities have parallel developments with, and are partly motivated by, recent interesting proposals which engage quantum correlations to such diverse tasks as e.g. quantum computation and communication [3, 4], quantum cryptography [5], teleportation [6], and some new frequency standards [7]. Although the classification of quantum correlations is still open to refinements, it appears to include the following cases: for pure states, correlations entail nonlocality and give rise to violation of Bell inequalities [2]. For mixed states, two systems are considered uncorrelated if the composite system density matrix factorizes into a product of reduced density matrices, one for each isolated quantum subsystem, namely $\rho = \rho_1 \otimes \rho_2$, where $\rho_{1,2} = \mathrm{tr}_{1,2}\,\rho$, are determined by partial tracing. Quantification measures for that case include the von Neumann entropy [8] and other invariant indices [9]. On the other hand classical correlations for quantum subsystems imply separability of the joint system density matrix, which is analysed into a convex sum for products of pure states namely, $\rho = \sum_i p_i \rho_1^i \otimes \rho_2^i$, $0 \leqslant p_i \leqslant 1$, $\sum_i p_i = 1$, [10]. Necessary and sufficient conditions for the existence of such convex decompositions for ρ's acting on $\mathbb{C}^2 \times \mathbb{C}^2$ and $\mathbb{C}^2 \times \mathbb{C}^3$ became available recently [11, 12]. Upper bounds for the number of terms in such convex expansions of separable matrices have also been determined, together with construction algorithms for the cases dim $\mathcal{H} \leqslant 6$ [13] and dim $\mathcal{H} \leqslant \infty$ [14]. Beyond

§ E-mail address: ellinas@aml.tuc.gr

‖ E-mail address: floratos@cyclades.nrcps.ariadne-t.gr

these types of classical correlations we encounter inseparable or entangled quantum states. For their characterization and the quantification of their entanglement some general conditions have been presented that good entanglement measures should satisfy [15].

In this letter we address the problem of the *prime decomposition* (PD) of a finite, but otherwise arbitrary, N-dimensional square density matrix ρ, into a sum of products of elementary density matrices, the number and the respective dimensions of which are determined by the compositeness of the dimension of ρ. This is achieved by means of (i) the so-called Chinese remainder theorem (CRT) [16], that is based on the prime decomposition of N (this also explains the name we have chosen for the decomposition), and (ii) by the fact that the discrete Heisenberg group H_N provides a projective representation of the abelian cyclic group $\mathbb{Z}_N$ [17]. More concretely, if $N = p_1^{m_1} p_2^{m_2} \dots p_t^{m_t}$ is the prime factor decomposition of N, where p's are distinct primes, then the PD of the density matrix involves square matrices $\rho^{(i)}$ $i = 1, \dots, t$, with power prime dimension equal to $N_i = p_i^{m_i}$. Also, the number t of ρ-factors is bounded by the number of coprime factors of N. As a measure of the correlation of a given mixed state ρ, with its possible prime or other decomposition, we evaluate the trace-norm distance between the two densities, study its unitary invariant symmetries, and interpret it in terms of the quantum variances between local operators of the subsystems of the decomposition.

We start by considering the matrix realization of the discrete Heisenberg group H_N generated by the operator set of N^2 elements $J_m \equiv J_{m_1 m_2} = \omega^{\frac{1}{2} m_1 m_2} g^{m_1} h^{m_2}$, where the matrices

$$g = \mathrm{diag}(1, \omega, \dots, \omega^{N-1}) \qquad h = \sum_{n \in \mathbb{Z}_N} |n\rangle\langle n+1| \tag{1}$$

satisfy the relations $\omega g h = h g$, $h^\dagger = h^{-1}$, $g^N = h^N = \mathbb{1}$, $h h^\dagger = h^\dagger h = \mathbb{1}$, $g g^\dagger = g^\dagger g = \mathbb{1}$, with $\omega^N = 1$, and $(m_1, m_2) \in \mathbb{Z}_N^2$, the square index-lattice. By virtue of these relations the following commutators are valid [18, 19, 20]:

$$[J_m, J_n] = -2i \sin\left[\frac{\pi}{N} m \times n\right] J_{m+n} \quad (\mathrm{mod}\ N). \tag{2}$$

Moreover, due to linear independence, completeness and the orthonormality issued by the relation

$$\mathrm{Tr}\, J_m J_n = N \delta_{m+n,0} \qquad (\mathrm{mod}\ N) \tag{3}$$

the same generator set forms a basis of the $su(N)$ matrices [17].

Let us consider an N-dimensional quantum system $\mathcal{S}$ with Hilbert space $\mathcal{H}_N$. The generators of the finite Heisenberg group H_N provide an operator basis $\{J_m | m \in \mathbb{Z}_N^2\}$ for the decomposition of the density matrix ρ of $\mathcal{S}$, namely

$$\rho = \frac{1}{N} \sum_{m \in \mathbb{Z}_N^2} (\lambda_m J_m) = \frac{1}{N}\left[\mathbb{1} + \sum_{m \in \mathbb{Z}_N^{*2}} \lambda_m J_m\right] \tag{4}$$

with $\mathbb{Z}_N^{*2} \equiv \mathbb{Z}_N \times \mathbb{Z}_N \setminus (0, 0)$. We note here that, due to the Hermitian conjugation of the basis elements i.e. $J_m^\dagger = J_{-m_1,-m_2} = J_{N-m_1,N-m_2}$, the Hermiticity of the density matrix implies for its elements the reality conditions $\lambda_m^* = \lambda_{N-m}$. Let us assume N to be a composite positive integer with prime-power decomposition $N = p_1^{n_1} p_2^{n_2} \dots p_t^{n_t} \equiv N_1 N_2 \dots N_t$, where each of the factors is distinct, uniquely determined and relatively prime to each other, i.e. $\gcd(N_i, N_j) = 1$ when $i \neq j$. Then according to CRT the isomorphism $\mathbb{Z}_N \cong \mathbb{Z}_{N_1} \oplus \cdots \oplus \mathbb{Z}_{N_t}$ is valid for the index-cyclic groups labelling the operator basis. To proceed we introduce the group isomorphic map $\mathbb{Z}_N^2 \xrightarrow{\delta} \mathbb{Z}_{N_1}^2 \oplus \cdots \oplus \mathbb{Z}_{N_t}^2$ between the cyclic groups. The explicit definition reads: $(m_1, m_2) \xrightarrow{\delta} (\delta(m_1), \delta(m_2)) \xrightarrow{\delta} (m_{11}; m_{21}, m_{12}; m_{22}, \ldots, m_{1t}; m_{2t})$, where $m_{1i} = m_1 - pN_i$, $m_{2i} = m_2 - qN_i$, $i = 1, \ldots, t$, $p, q \in \mathbb{Z}$, stand for the residues of the division of m_1, m_2 by N_i.

Next, we regard the fact that H_N provides a projective unitary representation of the additive cyclic group $\mathbb{Z}_N^2$, by means of the map $\mathbb{Z}_N^2 \xrightarrow{\pi_N} H_N$. More explicitly, $(m_1, m_2) \xrightarrow{\pi_N} \pi_N(m_1, m_2) = J_{m_1,m_2}$, with the property $\pi_N(m + n) = J_{m+n} = J_m J_n \mathrm{e}^{\frac{\mathrm{i}}{2} m\times n} = \pi_N(m)\pi_N(n)\mathrm{e}^{\frac{\mathrm{i}}{2} m\times n}$, where $m \times n := m_1 n_2 - m_2 n_1$. Then the following commutative diagram:

$$\begin{array}{cccccc} & \mathbb{Z}_N^2 & \xrightarrow{\delta} & \mathbb{Z}_{N_1}^2 \oplus \cdots \oplus \mathbb{Z}_{N_t}^2 & \\ \pi_N & \downarrow & & \downarrow & \pi_{N_1} \times \cdots \times \pi_{N_t} \\ & H_N & \xrightarrow[\pi_\delta]{} & H_{N_1} \otimes \cdots \otimes H_{N_t} & \end{array}$$

given by the equation $\pi_\delta \circ \pi_N = (\pi_{N_1} \times \cdots \times \pi_{N_t}) \circ \delta$, induces the isomorphism of CRT from the index-groups to the associated Heisenberg groups by the following component version of the above diagram:

$$\begin{array}{cccccc} & m & \xrightarrow{\delta} & \delta(m) = (m_{11}; m_{21}, \ldots, m_{1t}; m_{2t}) & \\ \pi_N & \downarrow & & \downarrow & \pi_{N_1} \times \cdots \times \pi_{N_t} \\ & \pi_N(m) = J_m & \xrightarrow[\pi_\delta]{} & \pi_\delta(J_m) = J_{\delta(m)} = J_{(m_{11}; m_{21}, \ldots, m_{1t}; m_{2t})} & \end{array}$$

We state the main proposition for the prime decomposition.

Proposition. *The isomorphism π_δ, determined by the commutative diagram figure, via its component version, is a linear map which induces the δ-map of CRT into the Heisenberg group H_N, and provides the unique PD of elements of H_N. Also π_δ is implemented by unitary operator in the Hilbert space $\mathcal{H}_N$ and possesses the co-associativity property.*

Proof. If $m \in \mathbb{Z}_N^2$ and $J_m = \omega^{1/2 m_1 m_2} g^{m_1} h^{m_2}$, then $\delta(m_1, m_2) = (m_{11}; m_{21}, \ldots, m_{1t}; m_{2t})$, with m_{1i} and m_{2i} the residues of the division of m_1, m_2 by N_i, respectively. According to CRT δ is an isomorphism, the determination of which provides the solution of a system of congruences $m_1 \equiv m_{1i}$ (mod N_i) and $m_2 \equiv m_{2i}$ (mod N_i), when $\gcd(N_i, N_j) = 1$, $N_i \neq N_j$, i.e. when N_i, N_j $i = 1, \ldots, t$, are pairwise coprime positive integers. Inversely, given the residues, the numbers m_1, m_2 can be determined in a mixed-radix notation by $m_1 \equiv \sum_{i=1}^t m_{1i} \overline{N}_i y_i$ and $m_2 \equiv \sum_{i=1}^t m_{2i} \overline{N}_i y_i$, (mod N), where $\overline{N}_i := \frac{N}{N_i}$ and y_i is the solution of the congruence $\overline{N}_i y_i \equiv 1$, (mod N_i). Alternatively, by means of the Euler function $\phi(k)$, which counts the positive integers $l \leqslant k$, which are coprime to k, the y_i are given by $y_i \equiv \overline{N}_i^{\phi(N_i)-1}$, (mod N_i). Then m_1, m_2, are expressed in the form $m_1 \equiv \sum_{i=1}^t m_{1i} \overline{N}_i^{\phi(N_i)}$ and $m_2 \equiv \sum_{i=1}^t m_{2i} \overline{N}_i^{\phi(N_i)}$ (mod N).

We turn now to study the consequences of this decomposition for the generators of H_N. With the notation as before we obtain the relations

$$g^{m_1} = g^{\sum_{i=1}^t m_{1i} \overline{N}_i^{\phi(N_i)}} = \prod_{i=1}^t g_i^{m_{1i}} \tag{5}$$

where $g_i := g^{\overline{N}_i \phi(N_i)}$ and $g_i^{N_i} = g^{\overline{N}_i N_i \phi(N_i)} = g^{N\phi(N_i)} = \mathbb{1}$. Analogous relations hold for the generator h^{m_2}. By direct computations it is verified that $g_i h_j = h_j g_i$ if $i \neq j$, and $g_i^k h_i^l = \omega_i^{kl} h_i^l g_i^k$, where $\omega_i := \omega^{\overline{N}_i^{2\phi(N_i)}}$. This definition implies that ω_i is periodic with respect to the coprime factors of N i.e. $\omega_i^{N_i} = \omega^{N_i \overline{N}_i^{2\phi(N_i)}} = 1$, for $i = 1, \ldots, t$. Using the above commutation properties of the generators we write:

$$\begin{aligned}\pi_\delta \circ \pi_N(m_1, m_2) = \pi_\delta(J_{m_1 m_2}) &= \prod_{i=1}^{t} \omega_i^{1/2 m_{1i} m_{2i}} g_i^{m_{1i}} h_i^{m_{2i}} \equiv \prod_{i=1}^{t} J^{(i)}_{m_{1i} m_{2i}} \\ &\cong \otimes_{i=1}^{t} J^{(i)}_{m_{1i} m_{2i}} = (\pi_{N_1} \times \cdots \pi_{N_t})(m_{11} m_{21}, \ldots, m_{1t} m_{2t}).\end{aligned} \tag{6}$$

The isomorphism introduced above is based on the fact that the $J^{(i)}_{m_{1i} m_{2i}}$'s are commuting for different i's and their moduli make them behave as copies (π_δ-isomorphic images) of the original $J_{m_1 m_2} \in H_N$, with periodicities $N_i \leqslant N$; this is similar to harmonics in Fourier analysis. The following embedding provides the explicit form of the isomorphism:

$$J^{(i)}_{m_{1i} m_{2i}} \cong \mathbb{1}_{N_1} \otimes \cdots \otimes J_{m_{1i} m_{2i}} \otimes \cdots \otimes \mathbb{1}_{N_t} = \pi_{N_i}(m_{1i}, m_{2i}) \in H_{N_i} \tag{7}$$

with $m_{1i}, m_{2i} \in \mathbb{Z}^2_{N_i}$; this provides the commutativity of the diagrams.

The prime decomposition of a general density matrix given in equation (4) with coefficients $\rho_m = \mathrm{Tr}_N(J_m \rho)$, can now be evaluated and reads,

$$\begin{aligned}\pi_\delta(\rho) &= \frac{1}{N} \sum_{(m_1,m_2) \in \mathbb{Z}_N^2} \rho_{\delta(m_1,m_2)} \pi_\delta(J_{m_1,m_2}) = \frac{1}{N} \sum_{(m_1,m_2) \in \mathbb{Z}_N^2} \rho_{\delta(m_1,m_2)} J_{\delta(m_1,m_2)} \\ &\cong \frac{1}{N} \sum_{(m_{11} m_{21}) \in \mathbb{Z}^2_{N_1}} \cdots \sum_{(m_{1t} m_{2t}) \in \mathbb{Z}^2_{N_t}} \rho_{m_{11} m_{21}, \ldots, m_{1t} m_{2t}} J_{m_{11} m_{21}} \otimes \cdots \otimes J_{m_{1t} m_{2t}}\end{aligned} \tag{8}$$

where $\rho_{m_{11} m_{21}, \ldots, m_{1t} m_{2t}} = \mathrm{Tr}_{N_1} \ldots \mathrm{Tr}_{N_t}(\pi_\delta(\rho) J_{m_{11} m_{21}} \otimes \cdots \otimes J_{m_{1t} m_{2t}})$.

This suggests that we first map ρ into $\pi_\delta(\rho)$, according to the previous analysis and then project along the $J^{(i)}_{m_{1i}\, m_{2i}}$'s , in order to determine the coefficients of the ρ-matrix factors in the prime decomposition.

A special form of PD that contains only a single product term is possible for a special class of density matrices with coefficients $\rho_m = \frac{1}{N} \omega^{f(m)}$, where $f(m) \in l_2(\mathbb{Z}_N^2)$, an arbitrary real function. If $f(m) = \sum_{kl} f_{kl} m_1^k m_2^l$, the transformed coefficients $\rho_{\delta(m)} = \frac{1}{N} \omega^{f(\delta(m))}$, due to $\omega^N = 1$, factorize as follows:

$$\begin{aligned}\rho_{\delta(m)} &= \frac{1}{N} \omega^{\sum_{i=1}^{t} \sum_{kl} f_{kl} m_{1i}^k m_{2j}^l \overline{N}_i^{\phi(N_i)} \overline{N}_j^{\phi(N_j)}} \\ &= \frac{1}{N} \omega^{\sum_{i=1}^{t} \sum_{kl} f_{kl} m_{1i}^k m_{2i}^l \overline{N}_i^{2\phi(N_i)}} \\ &= \prod_{i=1}^{t} \frac{1}{N_i} \omega_i^{f(m_{1i}, m_{2i})} =: \prod_{i=1}^{t} \rho_{(m_{1i}, m_{2i})}.\end{aligned} \tag{9}$$

Note that power raising is counted (mod N), so above a Frobenious type of map has been used i.e. $m_1^k = (\sum_{i=1}^{t} m_{1i} \overline{N}_i^{\phi(N_i)})^k = \sum_{i=1}^{t} m_{1i}^k \overline{N}_i^{\phi(N_i)}$ (mod N). Therefore we obtain the PD $\pi_\delta(\rho) = \rho^{(1)} \otimes \cdots \otimes \rho^{(t)}$, where $\rho^{(l)} = \sum_{(m_{1l}, m_{2l}) \in \mathbb{Z}^2_{N_l}} \rho_{(m_{1l}, m_{2l})} J_{m_{1l} m_{2l}}$. In view of the co-associative property of the π_δ, to be established shortly, we see that those matrices that admit such complete factorization behave as group-like elements under the PD map.

To proceed we study the unitary implementation of the PD of density matrices. We introduce the operator $V_\delta : \mathcal{H}_N \longrightarrow \otimes_{i=1}^{t} \mathcal{H}_{N_i}$, given by

$$V_\delta = \sum_{n \in \mathbb{Z}_N} |\delta(n)\rangle\langle n| \equiv \sum_{n \in \mathbb{Z}_N} |\{n_i\}\rangle\langle n| = \sum_{n \in \mathbb{Z}_N} |n_1\rangle \otimes \cdots \otimes |n_t\rangle\langle n| \tag{10}$$

and its conjugate

$$V_\delta^\dagger = V_{\delta^{-1}} = \sum_{\{n_i\} \in \{\mathbb{Z}_{N_i}\}} |\delta^{-1}(\{n_i\})\rangle\langle\{n_i\}| \equiv \sum_{\{n_i\} \in \{\mathbb{Z}_{N_i}\}} |\delta^{-1}(n_1, \ldots, n_t)\rangle\langle n_1| \otimes \cdots \otimes \langle n_t| \tag{11}$$

(this δ, as the one used earlier, maps numbers to their respective residues; see below). These operators form a conjugate pair that obeys the unitarity condition $V_\delta V_\delta^\dagger = \otimes_{i=1}^t \mathbb{1}_{\mathcal{H}_{N_i}}$ and $V_\delta^\dagger V_\delta = \mathbb{1}_{\mathcal{H}_N}$. It is then straightforward to verify that the PD map π_δ acting on general density matrix is implemented by the unitary similarity transformation i.e. $\pi_\delta(\rho) = V_\delta \rho V_\delta^\dagger \in \otimes_{i=1}^t H_{N_i}$.

Finally, we study briefly an important property of the PD map π_δ, namely that it becomes a co-associative comultiplication of the Heisenberg group H_N; this illustrates a connection of CRT with Hopf algebras [21] in the framework of quantum mechanical correlations. Consider the map $\delta_{n_1,n_2}(x) = (x - \rho n_1, x - \sigma n_2)$, $\rho, \sigma \in \mathbb{Z}$, by which a $x \in \mathbb{Z}_{n_1 n_2}$ decomposes into its residues w.r.t. coprimes n_1, n_2. Also consider its dual map $\mu_{n_1,n_2}(a, b) \equiv a n_2^{\phi(n_1)} + b n_1^{\phi(n_2)} = x \pmod{n_1 n_2}$, which constructs the solution of the congruences $x \equiv a \pmod{n_1}$, $x \equiv b \pmod{n_2}$, according to CRT. Then we check that for N_1, N_2, N_3, three coprime factors of N, the following equation is valid on any $a \in \mathbb{Z}_N$:

$$(\delta_{N_1,N_2} \times \mathrm{id}) \circ \delta_{N_1 N_2, N_3} = (\mathrm{id} \times \delta_{N_2,N_3}) \circ \delta_{N_1, N_2 N_3}. \tag{12}$$

This is dual to the relation

$$\mu_{N_1 N_2, N_3} \circ (\mu_{N_1,N_2} \times \mathrm{id}) = \mu_{N_1, N_2 N_3} \circ (\mathrm{id} \times \mu_{N_2,N_3}) \tag{13}$$

which holds if we are given three congruences and combine them pairwise in two different ways. This (co)associativity of the CRT maps, in turn is induced into the PD map π_δ, where it takes the form

$$(\pi_{\delta_{N_1,N_2}} \otimes \mathrm{id}) \circ \pi_{\delta_{N_1 N_2, N_3}} = (\mathrm{id} \otimes \pi_{\delta_{N_2,N_3}}) \circ \pi_{\delta_{N_1, N_2 N_3}}. \tag{14}$$

As an example we take the system

$$\begin{aligned} x &\equiv 2 \pmod{3} \\ x &\equiv 2 \pmod{4} \\ x &\equiv 3 \pmod{5} \end{aligned} \tag{15}$$

with solution $x = 38 \pmod{60}$, and obtain

$$\begin{aligned} &\mu_{3\cdot 4,5} \circ (\mu_{3,4} \times \mathrm{id})(2, 2, 3) = \mu_{3\cdot 4,5}(2, 3) = 38 \\ &\mu_{3,4\cdot 5} \circ (\mathrm{id} \times \mu_{4,5})(2, 2, 3) = \mu_{3,4\cdot 5}(2, 18) = 38. \end{aligned} \tag{16}$$

Dualizing, we recover the relation for the δ's which induces the co-associativity of the PD mapping:

$$(\mathrm{id} \otimes \pi_{\delta_{4,5}}) \circ \pi_{\delta_{3,4\cdot 5}}(\rho) = (\pi_{\delta_{3,4}} \otimes \mathrm{id}) \circ \pi_{\delta_{3\cdot 4,5}}(\rho) \tag{17}$$

for $\rho \in H_{60}$. Closing this proof we note that the integral $\int_N : H_N \longrightarrow \mathbb{C}$, with definition $\int_N \rho := \mathrm{Tr}_N\, \rho$, is invariant under the comultiplication $\pi_{\delta_{N_1,N_2}}$, in the sense that $(\int_{N_1} \otimes \int_{N_2}) \circ \pi_{\delta_{N_1,N_2}}(\rho) = \int_N \rho$. $\square$

We turn now to the study of the correlation between finite quantum systems. We start with two systems with state vector Hilbert spaces of dimension N_1, N_2 respectively. Any observable and density matrix is expressed by the elements of the Lie algebra $u(N_1)$, $u(N_2)$ correspondingly. For the density matrix of system-1 e.g.

$$\rho^{(1)} = \frac{1}{N_1}\left[\mathbb{1}^{(1)} + \sum_{m \in \mathbb{Z}_{N_1}^{2*}} \lambda_m^{(1)} J_m^{(1)}\right] \tag{18}$$

and similarly for system-2. The choice of the operator basis $(\mathbb{1}^{(i)}, J_m^{(i)})$, $(i = 1, 2)$, for the Lie algebra $u(N_i) \approx u(1) \oplus su(N_i)$ is an important one. For a composite system the density matrix reads [22]

$$\rho = \frac{1}{N_1 N_2}\Bigg[\mathbb{1}^{(1)} \otimes \mathbb{1}^{(2)} + \sum_{m\in\mathbb{Z}_{N_1}^{2*}} \lambda_m^{(1)} J_m^{(1)} \otimes \mathbb{1}^{(2)} + \sum_{n\in\mathbb{Z}_{N_2}^{2*}} \lambda_n^{(2)} \mathbb{1}^{(1)} \otimes J_n^{(2)} + \sum_{m\in\mathbb{Z}_{N_1}^{2*}} \sum_{n\in\mathbb{Z}_{N_2}^{2*}} \lambda_{mn}^{(1,2)} J_m^{(1)} \otimes J_n^{(2)}\Bigg] \tag{19}$$

where $\lambda_m^{(1)} \equiv \langle J_m^{(1)}\rangle = \mathrm{Tr}(\rho \cdot J_m^{(1)} \otimes \mathbb{1}^{(2)})$, $\lambda_m^{(2)} \equiv \langle J_m^{(2)}\rangle = \mathrm{Tr}(\rho \cdot \mathbb{1}^{(1)} \otimes J_m^{(2)})$ and $\lambda_{mn}^{(1,2)} \equiv \langle J_m^{(1)} \otimes J_n^{(2)}\rangle = \mathrm{Tr}(\rho \cdot J_m^{(1)} \otimes J_n^{(2)})$, the correlation components. Also by partial tracing we define $\rho^{(1)} = \mathrm{Tr}_2\, \rho$, $\rho^{(2)} = \mathrm{Tr}_1\, \rho$. To proceed with the definition of the correlation index we view the space of matrices $\rho \in u(N_1) \otimes u(N_2) \equiv \mathcal{G}$, as a norm space with Hilbert–Schmidt (HS) norm,

$$\|A\|_{(2)} \equiv \sqrt{< A, A >} = (\mathrm{Tr}\, A^\dagger A)^{1/2} = \sqrt{\sum_{ij=1}^{N^2} |a_{ij}|^2} \tag{20}$$

for $A = (a_{ij}) \in \mathcal{G}$. This is essentially a Frobenius-type matrix norm, which is unitarily invariant i.e. $\|UAY\| = \|A\|$, for U, Y unitary (the lower index of the norm will be omitted hereafter). Then we propose the following.

Definition. *The correlation scalar index of two coupled finite quantum systems in a mixed state ρ is defined as [22]*

$$\mathcal{E} \equiv \|\Delta\rho\|^2 = \|\rho - \rho^{(1)} \otimes \rho^{(2)}\|^2. \tag{21}$$

Index $\mathcal{E}$ provides us with a measure of correlation between the coupled systems in terms of the difference of the factorized partial density matrices from the density of the composite system. It is cast in the form

$$\begin{aligned}\mathcal{E} &= \|\rho\|^2 - 2\,\mathrm{Tr}(\rho \cdot \rho^{(1)} \otimes \rho^{(2)}) + \|\rho^{(1)}\|^2 \|\rho^{(2)}\|^2 \\ &= \frac{1}{N_1 N_2} \sum_{m\in\mathbb{Z}_{N_1}^{2*}} \sum_{n\in\mathbb{Z}_{N_2}^{2*}} [\lambda_{mn}^{(1,2)} - \lambda_m^{(1)}\lambda_n^{(2)}][\lambda_{N_1-m,N_2-n}^{(1,2)} - \lambda_{N_1-m}^{(1)}\lambda_{N_2-n}^{(2)}]\end{aligned} \tag{22}$$

where N_1, N_2-modulo arithmetic applies in the respective indices.

The index $\mathcal{E}$ vanishes for product states and by using the reality conditions of the λ_i's i.e. $\lambda_m^{(\nu)*} = \lambda_{m-N_\nu}^{(\nu)}$, $\nu = 1, 2$, $\lambda_{m,n}^{(1,2)*} = \lambda_{N_1-m,N_2-n}^{(1,2)}$ we introduce the matrix $\Lambda_{mn} := \lambda_{mn}^{(1,2)} - \lambda_m^{(1)}\lambda_n^{(2)} = \langle J_m^{(1)} \otimes J_n^{(2)}\rangle - \langle J_m^{(1)}\rangle\langle J_n^{(2)}\rangle$, and re-express the index in the form

$$\mathcal{E} = \frac{1}{N_1 N_2}\, \mathrm{Tr}\, \Lambda\Lambda^\dagger. \tag{23}$$

This last expression suggests first that the index $\mathcal{E}$ is determined by the trace of the covariance matrix of local observables $J_m^{(1)}$ and $J_n^{(2)}$, and second that it is invariant under general unitary tranformations of the group $U(N_1 \cdot N_2)$, i.e. $\Lambda \to \mathcal{U}^\dagger\Lambda\mathcal{U}$; $\Lambda^\dagger \to \mathcal{U}^\dagger\Lambda^\dagger\mathcal{U}$, with $\mathcal{U} \in U(N_1 \cdot N_2) \subset U(N_1) \otimes U(N_2)$. The last inclusion describes the fact that the invariance unitary group of $\mathcal{E}$ is in general larger than the local unitary transformations in which case the symmetry group factorizes (cf [9]).

Extensions to three and more coupled systems is straightforward. For three systems e.g. the composite density matrix involves terms of the operator basis where the J_m's are embedded in all possible ways in the 3-tensor space. Also for the reduced matrices there are various possibilities in this case i.e. $\rho^{i,j} = \mathrm{Tr}_k\, \rho$ and $\rho^i = \mathrm{Tr}_{jk}\, \rho$, with cyclic permutations of $(i, j, k) = (1, 2, 3)$. This gives rise to different correlation indices i.e.

$$\begin{aligned}
\mathcal{E}_{123} &= \|\rho - \rho^{(1)} \otimes \rho^{(2)} \otimes \rho^{(3)}\|^2_{(2)} \\
\mathcal{E}_{1(23)} &= \|\rho - \rho^{(1)} \otimes \rho^{(23)}\|^2_{(2)} \\
\mathcal{E}_{2(13)} &= \|\rho - \rho^{(2)} \otimes \rho^{(13)}\|^2_{(2)} \\
\mathcal{E}_{3(12)} &= \|\rho - \rho^{(3)} \otimes \rho^{(12)}\|^2_{(2)}.
\end{aligned} \tag{24}$$

Closing, we should mention that the correlation index can be expressed in terms of the P function of the involved density matrices, associated with the $SU(2)$ group coherent state of dimension N. This possibility, as will be explained elsewhere [23], is based on the fact the the $su(N)$ algebra generators used here in the expansion of the N-dimensional density matrices, can be embedded (by means of the polar decomposition of the $su(2)$ algebra), into the enveloping algebra $U(su(2))$. Examples of the finite case together with extensions to infinite-dimensional quantum systems will also be reported elsewhere.

We acknowledge support from the Greek Secretariat of Research and Technology under contract $\Pi ENE\Delta$ 95/1981.

References

[1] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[2] Bell J 1964 *Ann. Phys., NY* **1** 195
Clauser J F, Horne and Holt R A 1969 *Phys. Rev. Lett.* **23** 880
[3] Deutsch D 1989 *Proc. R. Soc.* **425** 73
[4] Bennett C H *et al* 1996 *Phys. Rev.* A **53** 2046
Deutsch D *et al* 1996 *Phys. Rev. Lett.* **77** 28
[5] Ekert A K 1991 *Phys. Rev. Lett.* **67** 661
[6] Bennett C H *et al* 1991 *Phys. Rev. Lett.* **67** 661
[7] Huelga S *et al* 1997 *Phys. Rev. Lett.* **79** 3865
[8] Barnett S M and Phoenix S J D 1991 *Phys. Rev.* A **44** 535
[9] Schlienz J and Mahler G 1995 *Phys. Rev.* A **52** 4396
[10] Werner R F 1989 *Phys. Rev.* A **40** 4277
[11] Peres A 1996 *Phys. Rev. Lett.* **77** 1413
[12] Horodecki M *et al* 1996 *Phys. Lett.* A **223** 1
[13] Sanpera A *et al* 1997 *Preprint* e-print archive quant-ph/9703004
[14] Lewenstein M and Sanpera A 1997 *Preprint* e-print archive quant-ph/9707043
[15] Vedral V *et al* 1997 *Phys. Rev. Lett.* **78** 2275
[16] Knuth D E 1981 *Seminumerical Algorithms* vol II (Reading, MA: Addison-Wesley)
[17] Weyl H 1950 *The theory of Groups and Quantum Mechanics* (New York: Dover) section 4.14
Schwinger J 1960 *Proc. Natl. Acad. Sci., USA* **47** 570
Schwinger J 1970 *Quantum Kinematics and Dynamics* (New York: Benjamin)
[18] Balian R and Itzykson C 1986 *C. R. Acad. Sci., Paris* **303** 773
[19] Fairlie D B *et al* 1989 *Phys. Lett.* B **218** 203
Fairlie D B *et al* 1989 *Phys. Lett.* B **224** 101
[20] Floratos E G 1989 *Phys. Lett.* B **228** 335
Floratos E G 1989 *Phys. Lett.* B **233** 395
Athanasiu G G *et al* 1994 *Nucl. Phys.* B **425** 343
Athanasiu G G *et al* 1996 *J. Phys. A: Math. Gen.* **29** 6737
[21] Abe E 1980 *Hopf Algebras* (Cambridge: Cambridge University Press)
[22] Ellinas D 1996 *Proc. IV Workshop on Physics and Computation* PhysComp96 ed T Toffoli *et al* (New England Complex Systems Institute) p 108
Ellinas D 1996 webpage http://pm.bu.edu/PhysComp96/
[23] Ellinas D and Floratos E G to appear